이학석사 학위논문

Long-term factorization of the stochastic discount factor via martingale extraction

(마팅게일 추출을 통한 확률할인인자의 장기적 분해)

2020년 8월

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이 논문을 이학석사 학위논문으로 제출함

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Long-term factorization of the stochastic discount factor via martingale extraction

A dissertation
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Abstract

This paper reviews the treatment of the long-term factorization of the stochastic discount factor in Qin and Linetsky (2017), and presents explicit forms of the long-term factorization in some concrete examples. The main purpose of this paper is to analyze relatively simple models in which the long-term factorization seems to be applied, so we restrict the semimartingale setting of Qin and Linetsky (2017) to rather simple one-dimensional time-homogeneous Markovian setting. Some other working models are also included as examples, though they have not been fully verified yet. We introduce and exploit the martingale extraction method, developed by Hansen and Scheinkman (2009), as the main tool for finding the long-term factorization.

Key words: Martingale extraction, long-term factorization, stochastic discount factor, positive eigenfunction, recurrent stochastic process
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Contents

Chapter 1

Introduction

Suppose that a market is free of arbitrage and frictionless. Then the stochastic discount factor(SDF) exists, and the present value of the future payoff is represented by the expectation of the product of the SDF and the future payoff. In that sense, the SDF is also referred to as the pricing kernel. One of the most familiar forms of the SDF process is $(e^{-\int_0^t r_s ds})_{t\geq 0}$, where r is a Markovian interest rate process. More generally, Hansen and Scheinkman (2009) introduced the concept of multiplicative functionals and used the concept to model the SDF in Markovian environments. Further, Qin and Linetsky (2017) defined the SDF as a strictly positive semimartingale without Markovian assumption.

There have been several researches studying long-term factorization of the SDF, and they utilized the long-term factorization for their needs. Alvarez and Jermann (2005) decomposed the SDF into the martingale part and the transitory component in discrete-time environments, and Hansen and Scheinkman (2009) studied positive eigenfunctions of pricing operators to factorize the SDF in Markovian environments. Finally Qin and Linetsky (2017) established the long-term factorization of the SDF in general semimartingale setting which unifies the works of Alvarez and Jermann (2005) and Hansen and Scheinkman (2009). They also identified their long-term factorization with the approach of Hansen and Scheinkman (2009) in

Markovian environments under some regularity assumptions.

The long-term factorization of the SDF process S in Qin and Linetsky (2017) is given by

$$S_t = e^{-\lambda t} \frac{1}{\pi_t} M_t^\infty,$$

where a process M^∞ is a postive martingale with $M_0^\infty = 1$ that defines the long-term forward measure, λ is a positive number which is considered as the long-term discount rate or yield on the long bond, and a process π is positive semimartingale with $\pi_0 = 1$ which characterizes holding preiod returns on the long bond net of the long-term discount rate. This interpretation seems clear if we observe the formula for the price of the zero-coupon bond $\mathcal{P}_t$ of maturity $t > 0$ under the long-term forward measure $\mathbb{Q}^\infty|_{\mathcal{F}_t} := M_t^\infty \mathbb{P}|_{\mathcal{F}_t}$, $t \geq 0$:

$$\mathcal{P}_t = \mathbb{E}^{\mathbb{P}}\left[S_t\right] = e^{-\lambda t} \mathbb{E}^{\mathbb{Q}^\infty}\left[1/\pi_t\right].$$

Qin and Linetsky (2017) defined the concepts of the long bond and the long forward measure by extending the zero-coupon bond valuation process for a given maturity T, say $(B_t^T)_{0 \leq t \leq T}$, to all $t \geq 0$ and taking $T \to \infty$ in some sense, and suggested some conditions that ensures the existence of the long bond and the long forward measure. They also identified the process π and M^∞ with the positive eigenfunction and the corresponding eigenmeasure of Hansen and Scheinkman (2009) satisfying some regularity conditions respectively, in Markovian environments. Therefore, we can exploit the method of Hasen-Scheinkman factorization in Markovian environments to find the long-term factorization proposed in Qin and Linetsky (2017). In fact, we follow the treatment of Hansen-Scheinkman factorization in Leung and Park (2017) rather than the original work of Hansen and Scheinkman (2009). Leung and Park (2017) named the method *martingale extraction*.

This paper aims to analyze the long-term factorization of the SDF in some one-dimensional time-homogeneous Markov diffusion models, including CIR model and 3/2 model. This paper is organized as follows. In Chapter 2, we introduce the method of martingale extraction as the main tool for finding the long-term factorization. We also briefly introduce the notions of recurrence and ergodicity, which are important concepts when establishing the relationship between the factorization in Qin and Linetsky (2017) and the work of Hansen and Scheinkman (2009). In Chapter 3, we calculate the price of the zero-coupon bond in the CIR model and in the 3/2 model as simple applications of martingale extraction. The results are used in Section 4.3. In chapter 4, we review the key results of Qin and Linetsky (2017) in one-dimensional time-homogeneous Markovian setting, and conduct the factorization in the CIR model and in the 3/2 model. Since the distributions of two models are well-known, it is possible to find explicit forms of the factorization. In addition, some further models that the verifications are still ongoing are presented with partial results in the last section of the chapter.

Chapter 2

Main tools and concepts

In this chapter, we introduce some main tools and concepts needed to investigate the long-term factorization in Chapter 4.

2.1 Notations, assumptions, and restrictions

First of all, we specify some notations used in this paper and restrict the class of stochastic processes we deal with. We also impose some assumptions to avoid technical issues that may happen.

Notation 1. Throughout this paper, the following notations will be used without any mention.

- $(\Omega, \mathcal{F})$: a measurable space. We call Ω the sample space and $\mathcal{F}$ the σ-algebra of events.
- $(\mathcal{F}_t)_{t\geq 0}$: a filtration. That is, a family of sub σ-algebras of $\mathcal{F}$ indexed by $t \in [0, \infty)$ satisfying $\mathcal{F}_s \subset \mathcal{F}_t$ for $0 \leq s < t < \infty$. All the stochastic processes in this paper are considered to be adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.
- B : a standard Brownian motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

- I : an open interval $I = (l, r)$, $-\infty \leq l < r \leq \infty$ in $\mathbb{R}$.
- SDE : It is just an abbreviation of the term 'stochastic differential equation'.

Restriction 1. In this paper, unless otherwise stated, for a stochastic process we only treat a one-dimensional time-homogeneous Markov diffusion with distributions $(\mathbb{P}_x)_{x\in I}$(that is, $\mathbb{P}_x(X_0 = x) = 1$ for each $x \in I$), satisfying the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \quad X_0 = \xi \in I \tag{2.1}$$

where $b, \sigma : I \to \mathbb{R}$ are continuously differentiable functions and ξ is deterministic.

Assumption 1. We assume that for each $x \in I$, the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ satisfies the usual conditions, i.e. the σ-algebra $\mathcal{F}$ is complete, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, and $\mathcal{F}_0$ contains all the $\mathbb{P}_x$-null sets.

Restriction 2. The stochastic discount factor S in this paper is of the form

$$S_t = e^{-\int_0^t r(X_s)ds}, \quad t \geq 0,$$

where the underlying process X is a solution of (2.1) with $I = (0, \infty)$, and r is a measurable function such that $r(x) > 0$ whenever $x > 0$.

Notation 2. We denote $\mathbb{E}_x^{\mathbb{P}}$ as the expectation with respect to $\mathbb{P}_x$ for each $x \in I$. We let $\mathbb{P}$ represent the whole family of measures $(\mathbb{P}_x)_{x\in I}$.

2.2 Martingale extraction

In this section, we introduce the martingale extraction method, which is the main tool of this paper. For terminologies and concepts introduced in this section, we refer to Leung and Park (2017).

Assumption 2. The SDE (2.1) has a unique strong solution for given any deterministic initial value $\xi \in I$.

For a continuously differentiable function r, put

$$\mathcal{L} := b\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} - r\,\cdot, \tag{2.2}$$

i.e. $\mathcal{L}$ is the infinitesimal generator of X with killing rate r. Then it is well-known that for any function $\pi \in C^2(\mathbb{R})$ the process

$$e^{-\int_0^t r(X_s)ds}\pi(X_t) - \int_0^t \mathcal{L}\pi(X_s)ds$$

is a local martingale. Suppose that there exist a real number λ and a positive function $\pi \in C^2(\mathbb{R})$ satisfying

$$\mathcal{L}\pi = -\lambda\pi. \tag{2.3}$$

Then the π is an element of the kernel of the infinitesimal generator $\mathcal{L}_\lambda$ of X with killing rate $r(x) - \lambda$, i.e. $\mathcal{L}_\lambda\pi = (\mathcal{L} + \lambda)\pi = 0$, so a process

$$e^{\lambda t-\int_0^t r(X_s)ds}\pi(X_t)$$

is a positive local martingale. Define a stochastic process M as

$$M_t := e^{\lambda t-\int_0^t r(X_s)ds}\frac{\pi(X_t)}{\pi(X_0)}, \quad t \geq 0. \tag{2.4}$$

Then the process M is a positive local martingale with $M_0 = 1$, and in differential form it is written as

$$dM_t = e^{\lambda t-\int_0^t r(X_s)ds}\pi(X_0)^{-1}(\sigma\pi')(X_t)dB_t. \tag{2.5}$$

If, in addition, M_t is a martingale, one can take advantage of a new measure $\mathbb{Q}^\pi$ defined on each $\mathcal{F}_t$, $t \geq 0$, as

$$\mathbb{Q}^\pi(A) := \int_A M_t\,d\mathbb{P}, \qquad A \in \mathcal{F}_t. \tag{2.6}$$

The probability measure $\mathbb{Q}^\pi$ is called the eigenmeausure with respect to π.

For a bounded measurable function $h : \mathbb{R} \to \mathbb{R}$ consider the expectation of the form

$$\mathbb{E}^{\mathbb{P}}[e^{-\int_0^t r(X_s)\,ds} h(X_t)]\,.$$

By the change of measure formula and (2.4) we have

$$\mathbb{E}^{\mathbb{P}}[e^{-\int_0^t r(X_s)ds} h(X_t)] = \mathbb{E}^{\mathbb{P}}\left[\frac{h(X_t)}{\pi(X_t)} M_t\right] e^{-\lambda t}\pi(X_0) \tag{2.7}$$

$$= \mathbb{E}^{\mathbb{Q}^\pi}\left[\frac{h(X_t)}{\pi(X_t)}\right] e^{-\lambda t}\pi(X_0). \tag{2.8}$$

One can see that the left hand side of (2.7) is path-dependent, whereas (2.8) depends only on X_t at time t. Thus, dealing with (2.8) instead of (2.7) is in general more tractible.

Definition 2.2.1. Let (λ, π) be an eigenpair of $\mathcal{L}$, that is, a pair (λ, π) satisfying (2.3). If the stochastic process M in (2.4) is a martingale, the pair (λ, π) is called an admissible eigenpair and we say that the pair (λ, π) admits the martingale extraction of $e^{-\int_0^t r(X_s)ds}$.

One can rewrite (2.5) as

$$\frac{dM_t}{M_t} = \left(\frac{\sigma\pi'}{\pi}\right)(X_t)dB_t.$$

By the Girsanov theorem, a stochastic process B^π defined as

$$B_t^\pi := B_t - \int_0^t \left(\frac{\sigma\pi'}{\pi}\right)(X_s)ds, \qquad t \geq 0,$$

is a standard Brownian motion under $\mathbb{Q}^\pi$, and the dynamics of the process X under $\mathbb{Q}^\pi$ is written in the form

$$dX_t = \left(b + \frac{\sigma^2\pi'}{\pi}\right)(X_t)dt + \sigma(X_t)dB_t^\pi.$$

From the construction of Itô integral, one can see that a local martingale $\int_0^\cdot \sigma_s \, dB_s$ where σ is a progressively measurable process is a martingale if

$$\mathbb{E}\left[\int_0^t \sigma_s^2 ds\right] < \infty$$

holds for all $t \geq 0$. There are also several criteria for a local martingale to be a true martingale. We state one of the criteria that will be useful later in advance.

Proposition 2.2.2. *If a local martingale M satisfies*

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |M_s|\right] < \infty$$

for each $t \geq 0$, then M is a true martingale.

To prove the Proposition 2.2.2 it sufficies to prove the following lemma.

Lemma 2.2.3. *Let $\mathscr{T}_t$ be the set of all stopping times τ such that $\tau \leq t$ a.s. for a given $t > 0$ and M be a local martingale. Suppose that $(M_\tau)_{\tau \in \mathscr{T}_t}$ is uniformly integrable for each $t > 0$. Then M is a martingale.*

Proof. By assumption $\mathbb{E}[|M_t|] < \infty$ for each $t \geq 0$. Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence of M, i.e. $\tau_n \leq \tau_{n+1}$ for each $n \geq 1$, $\lim_{n \to \infty} \tau_n = \infty$, and each stopped process M^{τ_n} is a martingale. Then for each $t > 0$, $(\tau_n \wedge t)_{n \in \mathbb{N}} \subset \mathscr{T}_t$. Since $(M_{\tau_n \wedge t})_{n \in \mathbb{N}}$ is uniformly integrable and $M_{\tau_n \wedge t} \to M_t$ a.s. as $n \to \infty$, $M_{\tau_n \wedge t} \to M_t$ in L^1 also holds(Klenke, 2014, Theorem 6.25). Thus,

$$\mathbb{E}[M_t | \mathcal{F}_s] = \lim_{n \to \infty} \mathbb{E}[M_{\tau_n \wedge t} | \mathcal{F}_s] = \lim_{n \to \infty} M_{\tau_n \wedge s} = M_s. \quad \text{(limit in } L^1\text{)}$$

for each $s, t \in \mathbb{R}$ satisfying $0 \leq s < t$. □

There is a strong criterion which can be used when we want to show that a local martingale of the form (2.4) is actually a true martingale. The following proposition is an adjustment to Pinsky, 1995, Chapter 4, Theorem 8.5 (ii).

Proposition 2.2.4. *Consider the operator*

$$L = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + V(x)$$

where $\sigma^2 > 0$ *on* I*,* $(\sigma^2)', b'$ *and* V *are locally Hölder continuous on* I*, that is, Hölder continuous on every compact subset of* I*. Let* π *be a positive twice continuously differentiable function satisfying* $L\pi = 0$*. Then*

$$\mathbb{E}_x^{\mathbb{P}}\left[e^{\int_0^t V(X_s)ds}\pi(X_t)\right] = \pi(x), \quad \forall t \geq 0, \forall x \in I,$$

if and only if a weak solution of the SDE

$$dX_t = \left(b + \sigma^2\frac{\pi'}{\pi}\right)(X_t)dt + \sigma(X_t)dB_t$$

does not explode.

Remark 2.1. Since b and σ are continuous, a local weak solution up to explosion time always exists(Ikeda and Watanabe, 1989, Chapter 4, Theorem 2.3). To show the non-explosion of a weak solution of a given model, we may use Proposition 2.3.4 in the next section.

Corollary 2.2.5. *Suppose that the operator* (2.2) *satisfies the conditions in Proposition 2.2.4 and there exists a positive twice continuously differentiable function* π *satisfying* (2.3)*. If a weak solution of the SDE*

$$dX_t = \left(b + \sigma^2\frac{\pi'}{\pi}\right)(X_t)dt + \sigma(X_t)dB_t$$

does not explode, then a local martingale (2.4) *is a true martingale.*

Proof. Put $V(x) = -r(X) + \lambda$. Then the operator

$$\mathcal{L}_\lambda = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + V(x)$$

satisfies the conditions in Proposition 2.2.4 and $\mathcal{L}_\lambda \pi = 0$. For $0 \leq t \leq T < \infty$, by Proposition 2.2.4 and by the markov property we have

$$\mathbb{E}^{\mathbb{P}}\left[e^{\lambda T - \int_0^T r(X_s)ds}\pi(X_T)\middle|\mathcal{F}_t\right] = e^{\lambda t - \int_0^t r(X_s)ds}\mathbb{E}^{\mathbb{P}}_{X_t}\left[e^{\lambda(T-t) - \int_0^{T-t} r(X_s)ds}\pi(X_{T-t})\right]$$

$$= e^{\lambda t - \int_0^t r(X_s)ds}\pi(X_t).$$

□

The use of this powerful result of Proposition 2.2.4 and Corollary 2.2.5, though, would be overkill. So we will try to use rather simple arguments if possible.

2.3 Recurrence and Ergodicity

Recurrence and Ergodicity are important concepts in the theory of diffusion processes. There are lots of contents related to recurrence and ergodicity, which can be found in many books and papers. We introduce only a tiny part among the huge contents that is closely related to the purpose of this paper. We follow Qin and Linetsky (2016) for recurrence definitions.

Definition 2.3.1 (Recurrence of a stochastic process)**.** Let X be a stochastic process. For a Borel set $B \in \mathscr{B}(\mathbb{R})$ and $x \in I$ define a measure

$$R(x, B) := \int_0^\infty \mathbb{P}_x(X_t \in B)dt.$$

We say that the process X is recurrent if for each $B \in \mathscr{B}(\mathbb{R})$ either $R(x, B) = 0$ or $R(x, B) = \infty$ for all $x \in I$.

Remark 2.2. The meausure $R(x, \cdot)$ is called Green's measure or potential measure of X. The interpetation of $R(x, \cdot)$ is explained just before Definition 3.1 in Qin and Linetsky (2016).

Definition 2.3.2. Let X be a stochastic process. An eigenpair (λ, π) satisfying (2.3) is called a *recurrent eigenpair* if X is recurrent under the eigenmeasure $\mathbb{Q}^\pi$ (2.6). In this case an eigenmeasure $\mathbb{Q}^\pi$ is called a recurrent eigenmeasure correspoding to π.

In fact, Qin and Linetsky (2016) gave more general definition of recurrence: recurrence of a Borel right process(Qin and Linetsky, 2016, Appendix A). In the case of (2.1), there is an equivalent statement of Definition 2.3.1.

Proposition 2.3.3. *Let X be a one-dimensional diffusion process. Then the process X is recurrent if and only if* $\mathbb{P}_x\left(\exists t \in [0, \infty) \text{ such that } X_t = y\right) = 1$ *for all* $x, y \in I$.

The proof can be found in Qin and Linetsky, 2016, Appendix B.

We introduce a useful tool for proving recurrence and non-explosion of a given stochastic process. We recall some auxiliary functions in advance. Fix $c \in I$. The scale density and the scale function are defined as

$$s(x) = e^{-2\int_c^x \frac{b(y)}{\sigma^2(y)} dy}$$

and

$$S(x) = \int_c^x s(y) dy, \tag{2.9}$$

repectively. For a weak solution of (2.1) we define the exit time from I:

$$\tau_I := \inf\left\{t \geq 0 : X_t \notin I\right\}. \tag{2.10}$$

Thus $\tau_I = \infty$, a.s. is equivalent to the non-explosion of the weak solution in I. The following proposition is part of Karatzas and Shreve, 1998, Chapter 5, Proposition 5.22.

Proposition 2.3.4. *Let X be a weak solution of the SDE* (2.1) *in I. Assume that the coefficients* $\sigma : I \to \mathbb{R}, b : I \to \mathbb{R}$ *satisfy*

$$\sigma^2(x) > 0, \forall x \in I, \tag{2.11}$$

$$\forall x \in I, \exists \epsilon > 0 \quad \textit{such that} \quad \int_{x-\epsilon}^{x+\epsilon} \frac{1+|b(y)|}{\sigma^2(y)} dy < \infty. \tag{2.12}$$

Let S and τ_I be given by (2.9) *and* (2.10)*, respectively. If $S(l+) = -\infty$ and $S(r-) = \infty$ for any fixed $c \in I$, then*

$$\mathbb{P}(\tau_I = \infty) = \mathbb{P}\left(\sup_{0 \le t < \infty} X_t = r\right) = \mathbb{P}\left(\inf_{0 \le t < \infty} X_t = l\right) = 1.$$

In particular, X is recurrent in the sense of Proposition 2.3.3.

From now on we briefly introduce the definintion of an invariant measure for the transition semigroup of X and an ergodic theorem for the transition semigroup. Here we denote the transition semigroup of X as $(T_t)_{t \ge 0}$, i.e.

$$T_t f(x) = \mathbb{E}_x^{\mathbb{P}}[f(X_t)],$$

for every $t \ge 0$ and for any bounded measurable function f.

Definition 2.3.5. A σ-finite Borel measure Π on I is called an invariant measure for the transition semigroup $(T_t)_{t \ge 0}$ if

$$\int_I T_t f(x) \Pi(dx) = \int_I f(x) \Pi(dx)$$

holds for every $t \ge 0$ and for any nonnegative function $f \in C_c(I)$. If $\Pi(I) = 1$, then we call Π an invariant probability measure.

If the process X is recurrent, then we can find an invariant measure in the explicit form. The speed density is defined as

$$m(x) = \frac{2}{\sigma^2(x) s(x)},$$

and we call $m(dx) = m(x)dx$ the speed measure. The following proposition is part of Lemma 20.19 in Kallenberg (1997).

Proposition 2.3.6. *If the process X is positive recurrent, then the invariant measure of the transition semigroup of X is given by the speed measure $m(x)dx$, up to constant multiplcation.*

The next theorem can be found in Kunita, 1997, Theorem 1.3.10.

Theorem 2.3.7 (Ergodic theorem for a transition semigroup)**.** *Suppose that the transition probability has a strictly positive continuous density, i.e. there is a σ-finite Borel measure μ supported by I and a function $p \in C((0,\infty) \times I^2)$ such that*

$$T_t f(x) = \int_I f(y) p(t,x,y) \mu(dy)$$

hold for all $x \in I$ and for any nonnegative function $f \in C_c(I)$. If X is recurrent and has an invariant probability measure Π, then

$$T_t f(x) \longrightarrow \int_I f \, d\Pi \qquad \text{as } t \to \infty$$

holds for all $x \in I$ and for any bounded measurable function f.

We note by Proposition 2.3.6 and Theorem 2.3.7 that if the assumptions in Theorem 2.3.7 hold, then

$$T_t f(x) \longrightarrow \int_I f(y) \frac{m(y)}{m(I)} dy \qquad \text{as } t \to \infty$$

holds for all $x \in I$ and for any bounded measurable function f.

Chapter 3

Pricing zero-coupon bond

In this chapter, we make use of martingale extraction to calculate the price of the zero-coupon bond in some interest rate models under which we can get the explicit values. The content of this chapter is just a simple application of martingale extraction, though, the results will be used in chapter 4.

3.1 Cox, Ingersoll and Ross(CIR) model

A Cox, Ingersoll and Ross(CIR) process X is a solution of the SDE

$$dX_t = (b - aX_t)dt + \sigma\sqrt{X_t}dB_t, \quad X_0 = \xi, \tag{3.1}$$

where $b, a, \sigma > 0$, $2b > \sigma^2$, and $\xi > 0$. Since linear growth condition ensures weak existence(Ikeda and Watanabe, 1989, Chapter 4, Theorem 2.4) and pathwise uniqueness holds(Yamada and Watanabe, 1971, Theorem 1), the SDE (3.1) has a unique strong solution(Ikeda and Watanabe, 1989, Chapter 4, Theorem 1.1).

Using Proposition 2.3.4, it can be shown that X stays positive under the conditions $2b > \sigma^2$ and $\xi > 0$. Indeed, (2.11) and (2.12) are clear, and for $I = (0, \infty)$ and fixed $c \in I$

$$\begin{aligned} S(x) &= \int_c^x \exp\left\{-2\int_c^y \frac{b - a\zeta}{\sigma^2\zeta} d\zeta\right\} dy \\ &= \int_c^x y^{-\frac{2b}{\sigma^2}} \exp\left\{\frac{2a}{\sigma^2} y\right\} dy. \end{aligned}$$

Thus $\lim_{x\to\infty} S(x) = \infty$ is trivial, and the condition $2b > \sigma^2$ forces $S(0+)$ to be $-\infty$.

It is well-known that the CIR model is an affine term structure model, i.e. the zero-coupon bond price can be written in the form

$$\mathcal{P}_{t,T} = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T X_s\,ds}\middle|\mathcal{F}_t\right] = A(t,T)e^{-B(t,T)X_t} \tag{3.2}$$

(Brigo and Mercurio, 2006), and in this case

$$\begin{aligned} A(t,T) &= \left(\frac{2\alpha \exp\{(a+\alpha)(T-t)/2\}}{2\alpha + (a+\alpha)(\exp\{\alpha(T-t)-1\})}\right)^{2b/\sigma^2}, \\ B(t,T) &= \frac{2(\exp\{\alpha(T-t)-1\})}{2\alpha + (a+\alpha)(\exp\{\alpha(T-t)-1\})}, \end{aligned} \tag{3.3}$$

where $\alpha = \sqrt{a^2 + 2\sigma^2}$.

Here we calculate the present value of the zero-coupon bond of maturity T in the CIR interest rate model via martingale extraction. The equation (2.3) with killing rate $r(x) = x$ is

$$\frac{1}{2}\sigma^2 x\pi''(x) + (b - ax)\pi'(x) - x\pi(x) = -\lambda\pi(x).$$

By straightforward calculation we get an eigenpair

$$(\lambda, \pi(x)) := (b\eta, e^{-\eta x}), \quad x > 0, \tag{3.4}$$

where

$$\eta = \frac{-a + \sqrt{a^2 + 2\sigma^2}}{\sigma^2}.$$

A process

$$M_t := e^{\lambda t - \int_0^t X_s\, ds - \eta(X_t - \xi)},\ t \geq 0$$

is then a local martingale, as we saw in section 2.2. Moreover, since X takes values in $(0, \infty)$ and $\eta > 0$,

$$\mathbb{E}_\xi^{\mathbb{P}}\left[\sup_{0 \leq s \leq t} |M_s|\right] \leq \mathbb{E}_\xi^{\mathbb{P}}\left[e^{\lambda t + \eta \xi}\right] < \infty, \quad \forall t \geq 0.$$

Thus, it is a true martingale by Propostion 2.2.2. Therefore, the pair (3.4) admits the martingale extraction of $e^{-\int_0^t X_s\, ds}$. Under the corresponding eigenmeasure $\mathbb{Q}^\pi$, the process

$$B_t^{\mathbb{Q}^\pi} = B_t + \sigma\eta \int_0^t X_s\, ds,\ t \geq 0$$

is a Brownian motion, and X follows

$$dX_t = (b - \alpha X_t)\, dt + \sigma\sqrt{X_t} dB_t^{\mathbb{Q}^\pi},$$

where $\alpha = \sqrt{a^2 + 2\sigma^2}$. Since only a is replaced by $\alpha > 0$ and the other coefficients remain the same compared with (3.1), X is recurrent under $\mathbb{Q}^\pi$, i.e. the eigenpair (3.4) is recurrent.

Using martingale extraction, one can see that the price of the zero-coupon bond of maturity T at the present time in the CIR model is given by

$$\mathcal{P}_T = \mathbb{E}_\xi^{\mathbb{P}}\left[e^{-\int_0^T r(X_s) ds}\right] = e^{-\lambda T - \eta\xi}\mathbb{E}_\xi^{\mathbb{Q}^\pi}\left[e^{\eta X_T}\right].$$

We can use the moment generating function of a CIR process (A.7) in Appendix A.2 if $\eta < c$ holds, where $c = 2\alpha/(1 - e^{-\alpha T})\sigma^2$. This is trivial, because

$$c = \frac{2\alpha}{(1 - e^{\alpha T})\sigma^2} > \frac{\sqrt{a^2 + 2\sigma^2} + a}{\sigma^2} > \frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2} = \eta.$$

Thus, in the CIR model we have the explicit value of the price of the zero-coupon bond, which is given by

$$\mathcal{P}_T = e^{-(bT + \xi)\eta}\left(\frac{1}{1 - \eta/c}\right)^{2b/\sigma^2} \exp\left\{\frac{2ce^{-\alpha T}\xi\eta}{\sigma^2(1 - \eta/c)}\right\},$$

and one can easily check that this value coincides with the value (3.2) and (3.3).

3.2 3/2 model

We consider the SDE

$$dX_t = (b - aX_t)X_t\,dt + \sigma X_t^{3/2}\,dB_t\,,\ X_0 = \xi \tag{3.5}$$

where $a, b, \sigma > 0$ and $\xi > 0$. We derive this model from the CIR model. Define Y as the solution of a CIR process

$$dY_t = ((a + \sigma^2) - bY_t)dt - \sigma\sqrt{Y_t}dB_t, \quad Y_0 = \frac{1}{\xi} > 0. \tag{3.6}$$

with $b, \sigma > 0$ and $a > -\sigma^2$. Although $-\sigma < 0$, it causes no any problems because $-B$ is also a standard Brownian motion under $\mathbb{P}$. Put $X_t = 1/Y_t$, $t \geq 0$ and $f(y) = 1/y$, $y \in \mathbb{R}$. Applying Itô formula to $X_t = f(Y_t)$, we get (3.5). Therefore, (3.5) has the unique strong solution, X takes values in $(0, \infty)$, and is recurrent.

The equation (2.3) with killing rate $r(x) = x$ for the 3/2 model is

$$\frac{1}{2}\sigma^2 x^3 \pi''(x) + (b - ax)x\,\pi'(x) - x\,\pi(x) = -\lambda\pi(x).$$

One can show that a pair

$$(\lambda, \pi(x)) := (b\eta, x^{-\eta})\,,\ x > 0 \tag{3.7}$$

is an eigenpair where

$$\eta := \frac{\sqrt{(a + \sigma^2/2)^2 + 2\sigma^2} - (a + \sigma^2/2)}{\sigma^2}\,.$$

A process

$$M_t := e^{\lambda t - \int_0^t X_s ds}\left(\frac{X_t}{\xi}\right)^{-\eta}, t \geq 0 \tag{3.8}$$

is then a local martingale, even a martingale. It is not difficult but quite messy to show, so we prove this fact in Appendix C.1. Under the corresponding eigenmeasure $\mathbb{Q}^\pi$, the process

$$B_t^{\mathbb{Q}^\pi} = \sigma\eta \int_0^t \sqrt{X_s}\, ds + B_t\,,\ t \geq 0$$

is a Brownian motion, and X follows

$$dX_t = (b - \theta X_t)X_t\, dt + \sigma {X_t}^{3/2}\, dB_t^{\mathbb{Q}^\pi},$$

where $\theta = a + \sigma^2\eta$. Since $\eta > 0,\ \theta > -\sigma^2/2$ holds and hence X is recurrent under $\mathbb{Q}^\pi$. i.e. the eigenpair (3.7) is recurrent.

The price of zero-coupon bond in the 3/2 model is given by

$$\mathcal{P}_T = \mathbb{E}_\xi^{\mathbb{P}}\left[e^{-\int_0^T X_s\, ds}\right] = e^{-\lambda T}\xi^{-\eta}\mathbb{E}_\xi^{\mathbb{Q}^\pi}\left[X_T^\eta\right].$$

Since $\eta < 2\theta/\sigma^2 + 1\,(\theta = a + \sigma^2\eta)$, one can make use of the formula (A.9). We conclude that

$$\mathcal{P}_T = e^{-\lambda T}\xi^{-\eta}c^\eta e^{-u}\frac{1}{\Gamma(\eta)}\int_0^1 e^{uz}z^{p-\eta}(1-z)^{\eta-1}dz,$$

where

$$c = \frac{2b}{\sigma^2(1-e^{-bT})},\ u = \frac{c}{\xi}e^{-bT},\ \text{and } p = \frac{2\theta}{\sigma^2} + 1.$$

Chapter 4

The long-term factorization

In this chapter, we follow the presentation of Qin and Linetsky (2017). Firstly, We examine two main theorems, one for the general long-term factorization result and the other for the relationship between this result and the Hansen-Scheinkman factorization in Markovian environments, and then we give explicit verifications of the results in the CIR model and in the 3/2 model. Finally, we present some other models that the verifications are in progress.

4.1 Preliminaries

Before investigating the *long-term* concepts we briefly review the T-forward measure and change of numeraire and extend the T-forward measure to $\mathcal{F}_t$ for all $t \geq 0$. This useful tool when dealing with interest rate derivatives is the starting point for the main topic in Chapter 4.

Fix $T > 0$ and denote by $\mathcal{P}_{t,T}$ the price of the zero-coupon bond maturing at T, i.e.

$$\mathcal{P}_{t,T} = \mathbb{E}^{\mathbb{P}} \left[e^{-\int_t^T r(X_u)du} \middle| \mathcal{F}_t \right].$$

Recall that the T-forward measure $\mathbb{Q}^T$ on $\mathcal{F}_T$ is the equivalent martingale measure for the numeraire process $\mathcal{P}_{t,T}$, and

$$\mathbb{Q}^T|_{\mathcal{F}_T} = \frac{e^{-\int_0^T r(X_u)du}}{\mathcal{P}_T}\mathbb{P}|_{\mathcal{F}_T}.$$

The Radon-Nikodym derivative process $(M_t^T)_{0\le t\le T}$ is thus $M_t^T = e^{-\int_0^t r(X_u)du}\mathcal{P}_{t,T}/\mathcal{P}_T$. For $0 \le s < t \le T$ the time-s price Φ_s of the derivative with payoff Φ at t is given by

$$\Phi_s = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_s^t r(X_u)du}\Phi\middle|\mathcal{F}_s\right] = \mathcal{P}_{s,T}\mathbb{E}^{\mathbb{Q}^T}\left[\frac{\Phi}{\mathcal{P}_{t,T}}\middle|\mathcal{F}_s\right].$$

In particular,

$$\mathcal{P}_{s,t} = \mathcal{P}_{s,T}\mathbb{E}^{\mathbb{Q}^T}\left[\frac{1}{\mathcal{P}_{t,T}}\middle|\mathcal{F}_s\right].$$

The T-forward measure here is defined on $\mathcal{F}_t$ for $t \le T$, but we want to extend it to be defined on $\mathcal{F}_t$ for all $t > T$. To this end, we use a self-financing roll-over strategy as follows.

1. At time zero, invest one unit of account in the zero-coupon bond of maturity T. Then at time T we get $1/\mathcal{P}_T$ units of account, and at time $t \in [0,T)$ the value of our strategy B_t^T is equal to $\mathcal{P}_{t,T}/\mathcal{P}_T$.

2. Re-invest $1/\mathcal{P}_T$ units of account in the zero-coupon bond with maturity $2T$. Then at time $2T$ we get $1/(\mathcal{P}_T\mathcal{P}_{T,2T})$ units of account, and at time $t \in [T,2T)$ the valuation process of our self-financing strategy B_t^T is given by $B_t^T = \mathcal{P}_{t,2T}/\left(\mathcal{P}_T\mathcal{P}_{T,2T}\right)$.

3. Proceed the roll-over strategy. Then at time $t \in [kT,(k+1)T)$ for $k \in \mathbb{N}$ we have the valuation process

$$B_t^T = \frac{\mathcal{P}_{t,(k+1)T}}{\prod_{i=0}^k \mathcal{P}_{iT,(i+1)T}} \tag{4.1}$$

for our self-financing roll-over strategy.

For the process B^T define $M_t^T := e^{-\int_0^t r(X_s)ds} B_t^T$ for each $t \geq 0$. Then by construction M^T is a positive martingale, and it reduces to the Radon-Nikodym derivative process $e^{-\int_0^t r(X_u)du}\mathcal{P}_{t,T}/\mathcal{P}_T$ on $\mathcal{F}_T$. Thus, the process M^T is well-defined for all $t \geq 0$, and the T-forward measure $\mathbb{Q}^T$ are defined on $\mathcal{F}_t$ for all $t \geq 0$ as $\mathbb{Q}^T|_{\mathcal{F}_t} = M_t^T \mathbb{P}|_{\mathcal{F}_t}$, $\forall t \geq 0$. Finally, we can factorize the stochastic discount factor as

$$e^{-\int_0^t r(X_s)ds} = \frac{M_t^T}{B_t^T}$$

for all $t \geq 0$.

4.2 Main theorems

We begin this section with two underlying definitions.

Definition 4.2.1 (Long bond)**.** If the value processes (4.1) converges to a strictly positive semimartingale $(B_t^\infty)_{t\geq 0}$ uniformly on compacts in probability as $T \to \infty$, i.e. for each $t > 0$, $\sup_{0\leq s\leq t} |B_s^T - B_s^\infty| \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$, then we call the limit $(B_t^\infty)_{t\geq 0}$ the *long bond*.

For the uniform convergence on compacts in probability, we refer to Protter (2004).

Definition 4.2.2 (Long forward meausre)**.** A measure $\mathbb{Q}^\infty$ is called the *long forward measure* if it is locally equivalent to $\mathbb{P}$ (i.e. $\mathbb{Q}^\infty|_{\mathcal{F}_t} \approx \mathbb{P}|_{\mathcal{F}_t}$ for each $t \geq 0$) and

$$\lim_{T\to\infty} \mathbb{Q}^T(A) = \mathbb{Q}^\infty(A) \quad \text{for each } A \in \mathcal{F}_t \text{ and } t \geq 0.$$

If the long forward measure exists, we denote it as $\mathbb{L}$.

Theorem 3.1 in Qin and Linetsky (2017) suggests a sufficient condition(It is also important to this paper) that guarantees stronger modes of convergence of $(B_t^T)_{t\geq 0}$ and $\mathbb{Q}^T$ to $(B_t^\infty)_{t\geq 0}$ and $\mathbb{L}$ respectively. However, the content is not directly related to this paper, so we omit the statement.

We now state part of the long-term factorization theorem for the stochastic dicount factor and the long bond in Qin and Linetsky (2017). The results of this theorem are subject to verifications in specific models that appeared in Chapter 3.

Definition 4.2.3. We say a function $L : (0,\infty) \to (0,\infty)$ is *slowly varying* if L is measurable and

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1$$

for all $a > 0$.

Theorem 4.2.4 (Long-term factorization)**.** *Suppose that for each $t > 0$ the following assumptions hold:*

(1) For each $t > 0$, there exists a $\mathcal{F}_t$-measurable random variable M_t^∞ such that $M_t^\infty > 0$ a.s. and

$$\frac{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T r(X_s)ds}\middle|\mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T r(X_s)ds}\right]} \xrightarrow{L^1} M_t^\infty \text{ as } T \to \infty. \tag{4.2}$$

(2) The limit $\lim_{T\to\infty} \frac{\mathcal{P}_{T-t}}{\mathcal{P}_T}$ exists as a positive finite value.

Then the following results hold:

(i) There exists a constant λ such that

$$\lim_{T\to\infty} \frac{\mathcal{P}_{T-t}}{\mathcal{P}_T} = e^{\lambda t}$$

for all $t \geq 0$. For this λ, there exists a slowly varying function L such that

$$\mathcal{P}_t = e^{-\lambda t} L(e^t),\ t \geq 0. \tag{4.3}$$

(ii) The long bond $(B_t^\infty)_{t\geq 0}$ exists and the stochastic discount factor possesses the long-term factorization

$$e^{-\int_0^t r(X_s)ds} = \frac{M_t^\infty}{B_t^\infty}.$$

Furthermore, the long bond also possesses a factorization $B_t^\infty = e^{\lambda t}\pi_t$ *for some positive semimartingale* π_t *with* $\pi_0 = 1$*, and hence*

$$e^{-\int_0^t r(X_s)ds} = e^{-\lambda t}\frac{M_t^\infty}{\pi_t}.$$

(iii) The positive semimartingale π_t *satisfies*

$$\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r(X_s)ds}\pi_T \middle| \mathcal{F}_t\right] = e^{-\lambda(T-t)}\pi_t \tag{4.4}$$

for all $0 \le t < T$.

Remark 4.1. For $0 \le t < T$, we note that

$$M_t^T = e^{-\int_t^T r(X_s)ds}\frac{\mathcal{P}_{t,T}}{\mathcal{P}_T} = e^{-\int_t^T r(X_s)ds}\frac{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_t^T r(X_s)ds}\pi_T \middle| \mathcal{F}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T r(X_s)ds}\right]}$$

coincides with the ratio in (4.2). Also, (4.4) indicates that the process $\left(e^{\lambda t - \int_0^t r(X_s)ds}\pi_t\right)_{t\ge 0}$ is a martingale under $\mathbb{P}$.

Remark 4.2. Theorem 3.2 in Qin and Linetsky (2017) suggests an explicit sequence of semimartingales $(\pi_t^T)_{t\ge0}$ and a specific mode of convergence such that $\pi_t^T \to \pi_t$ in that sense. The content, however, is outside the scope of this context, so we omit the full details.

Let us impose the time-homogeneous markov diffusion condition on the underlying stochastic process X again. We introduce part of Theorem 4.2 in Qin and Linetsky (2017), which identifies Theorem 4.2.4 with the long-term factorization by Hansen and Scheinkman (2009) under additional assumptions.

Definition 4.2.5. Let $\mathscr{L}$ be the set of all measurable functions $f \in C^2(\mathbb{R})$ such that $f(X_t) \in L^1(\mathbb{P})$ for all $t \ge 0$. If, for each $t \ge 0$, we set

$$\mathscr{P}_t f(x) = \mathbb{E}_x^{\mathbb{P}}\left[e^{-\int_0^t r(X_s)ds} f(X_t)\right],$$

then $\mathscr{P}_t : \mathscr{L} \to \mathbb{R}$ becomes a well-defined operator on $\mathscr{L}$. We call the operator $\mathscr{P}_t$ the *pricing operator*. In particular, we denote the zero-coupon bond valuation process with initial state $x \in I$ as $\mathcal{P}_t(x)$, i.e.

$$\mathcal{P}_t(x) = \mathbb{E}_x^{\mathbb{P}}\left[e^{-\int_0^t r(X_s)ds}\right], \quad x \in I.$$

Suppose that for all $t \geq 0$ there exist a constant $\lambda \in \mathbb{R}$ and a positive function π satisfying

$$\mathscr{P}_t\pi(x) = e^{-\lambda t}\pi(x). \tag{4.5}$$

Hansen and Scheinkman (2009) factorized the stochastic discount factor into

$$e^{-\int_0^t r(X_s)ds} = e^{-\lambda t}\frac{\pi(X_0)}{\pi(X_t)}M_t^{\pi}.$$

Then M^{π} is a local martingale under $\mathbb{P}$. Indeed, if we rearrange the factorization for M_t^{π}, we have

$$M_t^{\pi} = e^{\lambda t-\int_0^t r(X_s)ds}\frac{\pi(X_t)}{\pi(X_0)}, \tag{4.6}$$

which is of the form (2.4) in Section 2.2. If (4.6) is a martingale, then there is a probability measure $\mathbb{Q}^{\pi}$ such that $\mathbb{Q}^{\pi}|_{\mathcal{F}_t} = M_t^{\pi}\mathbb{P}|_{\mathcal{F}_t}$ for each $t \geq 0$. From now on we assume that (4.6) is a true martingale.

Qin and Linetsky (2017) proved that if the condition (4.2) holds under $\mathbb{P}_{\xi}$ for each initial state $\xi \in I$ and if $\mathcal{P}_{T-t}(\zeta)/\mathcal{P}_t(\xi)$ converges to a positive limit as $T \to \infty$ for each $t > 0$ and $\xi, \zeta \in I$, then there exists a positive eigenfunction π of pricing operators which factorizes the long bond $(B_t^{\infty}(\xi))_{t\geq 0,\xi\in I}$ into

$$B_t^{\infty}(\xi) = e^{\lambda t}\frac{\pi(X_t)}{\pi(\xi)},$$

where λ is the correspoding eigenvalue, and they identified the long forward measure $\mathbb{L}$ with the corresponding eigenmeasure $\mathbb{Q}^{\pi}$. Positive eigenfunctions of pricing operators, however, are in general not unique. The following theorem suggests one way to single out such π among all the positive eigenfunctions of pricing operators.

Theorem 4.2.6. *Suppose that there exists a positive eigenfunction π_R satisfying* (4.5) *and let λ be the corresponding eigenvalue. If π_R is recurrent under the corresponding eigenmeasure $\mathbb{Q}^{\pi_R}$ and*

$$\frac{\mathbb{E}^{\mathbb{Q}^{\pi_R}}_{X_t}\left[\pi_R(X_{T-t})^{-1}\right]}{\mathbb{E}^{\mathbb{Q}^{\pi_R}}_{\xi}\left[\pi_R(X_T)^{-1}\right]} \xrightarrow{L^1} 1 \text{ as } T \to \infty \tag{4.7}$$

for every initial value $\xi \in I$, then the following holds:

(1) The condition (4.2) *holds with $M_t^\infty = M_t^{\pi_R}$.*

(2) The long bond $(B_t^\infty(\xi))_{t\geq 0}$ exists for every $\xi \in I$ and is factorized by

$$B_t^\infty(\xi) = e^{-\lambda t}\frac{\pi_R(X_t)}{\pi_R(\xi)}.$$

(3) The corresponding recurrent eigenmeasure $\mathbb{Q}^{\pi_R}$ coincides with the long forward measure $\mathbb{L}$.

Remark 4.3. The original statement in Qin and Linetsky (2017) of Theorem 4.2.6 requires $\mathscr{P}_t$ to have the exponential ergodicity as a sufficient condition. This is because they exploited a spectral theory when dealing with these kinds of problems, especially in Qin and Linetsky (2016). In the proof of Theorem 4.2.6, however, the exponential ergodicity is only used to prove the condition (4.7). Hence we replace the exponential ergodicity condition with (4.7).

Now it seems clear how to use the martingale extraction method for the long-term factorization. Indeed, for each $t \geq 0$ and $x \in I$ an eigenpair (2.3) is also the eigenpair of the pricing operator $\mathscr{P}_t$, namely

$$\begin{aligned}\mathscr{P}_t\pi(x) = \mathbb{E}^{\mathbb{P}}_x\left[e^{-\int_0^t r(X_s)ds}\pi(X_t)\right] &= e^{-\lambda t}\pi(x)\mathbb{E}^{\mathbb{P}}_x\left[e^{\lambda t-\int_0^t r(X_s)ds}\frac{\pi(X_t)}{\pi(x)}\right] \\ &= e^{-\lambda t}\pi(x).\end{aligned}$$

Therefore, an admissible eigenfunction which admits the martingale extraction of the stochastic discount factor $e^{-\int_0^t r(X_s)ds}$ is a positive eigenfunction of pricing operators.

4.3 Examples of models

In this section, we conduct direct calculations with the two models appeared in Chapter 3 to confirm that the long-term factorization in Theroem 4.2.4 via Theorem 4.2.6 works well. To this end, we shall see that the conditions in Theorem 4.2.6 and Theorem 4.2.4 hold, and then present explicit results. However, we shall avoid to show the convergence of the long bond explicitly, because we have not introduced the notion of semimartingale topology, which is beyond the scope of this paper.

4.3.1 CIR model

We recall that the CIR model is given by the solution of the SDE

$$dX_t = (b - aX_t)dt + \sigma\sqrt{X_t}dB_t, \quad X_0 = \xi,$$

and we imposed the conditions $b, a, \sigma > 0$, $2b > \sigma^2$, and deterministic initial value $\xi > 0$.

Conditions

Recurrence. The given conditions ensures that the eigenpair (3.4) is recurrent, as we saw in Section 3.1.

Condition (4.7). Recall that we used notations $c = 2\alpha/(1 - e^{-\alpha t})\sigma^2$ and $\alpha = \sqrt{a^2 + 2\sigma^2}$ in Section 3.1. For convenience, we consider c as a function of t, i.e.

$$c(t) = \frac{2\alpha}{(1 - e^{-\alpha t})}, \quad t \geq 0.$$

We have an explicit formula for $\mathbb{E}_\xi^\mathbb{Q}[1/\pi(X_{T-t})]$, which we have shown in Section 3.1:

$$\mathbb{E}_\xi^\mathbb{Q}\left[\frac{1}{\pi(X_{T-t})}\right] = \mathbb{E}_\xi^\mathbb{Q}[e^{\eta X_{T-t}}] = \left(\frac{1}{1 - \eta/c(T-t)}\right)^{2b/\sigma^2} \exp\left\{\frac{2c(T-t)e^{-\alpha(T-t)}\xi\eta}{\sigma^2(1 - \eta/c(T-t))}\right\}.$$

for each $0 \le t < T$. It is clear that

$$\lim_{T\to\infty} \mathbb{E}_{\xi}^{\mathbb{Q}}\left[\frac{1}{\pi(X_{T-t})}\right] = \left(\frac{1}{1-\sigma^2\eta/2\alpha}\right)^{2b/\sigma^2}, \tag{4.8}$$

whence

$$\frac{\mathbb{E}_{X_t(\omega)}^{\mathbb{Q}^\pi}\left[\pi(X_{T-t})^{-1}\right]}{\mathbb{E}_{\xi}^{\mathbb{Q}^\pi}\left[\pi(X_T)^{-1}\right]} \longrightarrow 1 \text{ as } T\to\infty \tag{4.9}$$

for each $\omega \in \Omega$. Now we prove that (4.9) also holds in L^1 sense. To this end, we should find a function in $L^1(\mathbb{Q}^\pi)$ dominating the ratio in (4.9). Since $\mathbb{E}_{\xi}^{\mathbb{Q}^\pi}\left[\pi(X_T)^{-1}\right]$ is positive for all $T > 0$ and converges to a positive value, for any fixed $T_0 > 0$

$$\inf_{T\ge T_0} \mathbb{E}_{\xi}^{\mathbb{Q}^\pi}\left[\pi(X_T)^{-1}\right] > 0$$

holds. Fix $t > 0$. There exists $T_1 > 0$ such that $T \ge T_1$ implies

$$\frac{2c(T-t)e^{-\alpha(T-t)}\eta}{\sigma^2(1-\eta/c(T-t))} \le \frac{c(t)}{2},$$

because the left hand side converges to zero as $T\to\infty$. Hence, we have

$$\mathbb{E}_{X_t}^{\mathbb{Q}^\pi}\left[\frac{1}{\pi(X_{T-t})}\right] \le \left(\frac{1}{1-\sigma^2\eta/2\alpha}\right)^{2b/\sigma^2} \exp\left\{\frac{c(t)}{2}X_t\right\}.$$

We note that $\exp\{c(t)X_t/2\} \in L^1(\mathbb{Q}^\pi)$(See (A.7)). Therefore, for each $t > 0$ there exist $T_1 > 0$ and a constant $C > 0$ such that $T \ge T_1$ implies

$$\frac{\mathbb{E}_{X_t}^{\mathbb{Q}^\pi}\left[\pi(X_{T-t})^{-1}\right]}{\mathbb{E}_{\xi}^{\mathbb{Q}^\pi}\left[\pi(X_T)^{-1}\right]} \le C\exp\left\{\frac{c(t)}{2}X_t\right\},$$

and we conclude that L^1-convergence also holds.

Condition (4.2) Using the markov property and the martingale extraction method, one can see that

$$\frac{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T X_s\,ds}\middle|\mathcal{F}_t\right]}{\mathbb{E}_{\xi}^{\mathbb{P}}\left[e^{-\int_0^T X_s\,ds}\right]} = \exp\left\{\lambda t - \int_0^t X_s\,ds - \eta(X_t-\xi)\right\}\frac{\mathbb{E}_{X_t}^{\mathbb{Q}^\pi}\left[\pi(X_{T-t})^{-1}\right]}{\mathbb{E}_{\xi}^{\mathbb{Q}^\pi}\left[\pi(X_T)^{-1}\right]}.$$

We already know that $(e^{\lambda t-\int_0^t X_s\,ds-\eta(X_t-\xi)})_{t\geq 0}$ is a positive $\mathbb{P}$-martingale and the Radon-Nikodym derivative $d\mathbb{Q}^\pi/d\mathbb{P}|_{\mathcal{F}_t}$ for each $t>0$. Moreover, by the condition (4.7) we know the ratio in the right hand converges to 1 in $L^1(\mathbb{Q}^\pi)$. Therefore,

$$\frac{\mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^T X_s\,ds}\middle|\mathcal{F}_t\right]}{\mathbb{E}_\xi^{\mathbb{P}}\left[e^{-\int_0^T X_s\,ds}\right]}\xrightarrow{L^1(\mathbb{P}_\xi)} e^{\lambda t-\int_0^t X_s\,ds-\eta(X_t-\xi)}\qquad\text{as } T\to\infty$$

and the condition holds.

Condition (2) This can be easily done through calculations similar to those in the condition (4.2).

$$\lim_{T\to\infty}\frac{\mathcal{P}_{T-t}(\xi)}{\mathcal{P}_T(\xi)}=\lim_{T\to\infty}\frac{\mathbb{E}_\xi^{\mathbb{P}}\left[e^{-\int_0^{T-t} X_s\,ds}\right]}{\mathbb{E}_\xi^{\mathbb{P}}\left[e^{-\int_0^T X_s\,ds}\right]}=\lim_{T\to\infty}e^{\lambda t}\frac{\mathbb{E}_\xi^{\mathbb{Q}^\pi}\left[\pi(X_{T-t})^{-1}\right]}{\mathbb{E}_\xi^{\mathbb{Q}^\pi}\left[\pi(X_T)^{-1}\right]}=e^{\lambda t}$$

for every $t>0$ and $\xi\in I$.

Remark 4.4. The speed density of the CIR process under $\mathbb{Q}^\pi$ is given by

$$m(x)=Cx^{2b/\sigma^2-1}e^{-(2\alpha/\sigma^2)x}$$

for some constant $C>0$. By Proposition 2.3.6,

$$\psi(y):=\frac{(2\alpha/\sigma^2)^{2b/\sigma^2}}{\Gamma(2b/\sigma^2)}y^{2b/\sigma^2-1}e^{-(2\alpha/\sigma^2)y}$$

is an invariant probability density of X. This is the density of a Gamma distribution $\Gamma(2b/\sigma^2,\sigma^2/2\alpha)$. One can easily see by the moment generating function of the Gamma distribution that

$$\int_0^\infty e^{\eta y}\psi(y)dy=\left(\frac{1}{1-\sigma^2\eta/2\alpha}\right)^{2b/\sigma^2},$$

which coincides with (4.8). This is indeed the result of Theorem 2.3.7. We cannot apply Theorem 2.3.7 directly, however, because $1/\pi(x)=e^{\eta x}$, $x>0$ is not bounded.

Results

The only remaining part of the results of Theorem 4.2.4 we have to verify is (4.3). Put

$$L(x) = e^{-\eta\xi}\left(1 - \frac{\sigma^2\eta}{2\alpha}\left(1 - \frac{1}{x^\alpha}\right)\right)^{-\frac{2b}{\sigma^2}} \exp\left\{\frac{4\alpha\xi}{(x^\alpha - 1)\sigma^2}\Big/\sigma^2\left(1 - \frac{\sigma^2\eta}{2\alpha}\left(1 - \frac{1}{x^\alpha}\right)\right)\right\}.$$

Then for any $a > 0$, $\lim_{x\to\infty} L(ax) = e^{-\eta\xi}(1 - \sigma^2\eta/2\alpha)^{-2b/\sigma^2}$. Hence, L is a slowly varying function. The function L actually satisfies $L(e^t) = e^{\lambda t}\mathcal{P}_t$.

4.3.2 3/2 model

We recall that the 3/2 model is given by the solution of the SDE

$$dX_t = (b - aX_t)X_t dt + \sigma X_t^{3/2} dB_t, \quad X_0 = \xi,$$

and we imposed the conditions $b, a, \sigma > 0$ and deterministic initial value $\xi > 0$.

Conditions

Recurrence. We saw in Section 3.2 that the eigenpair (3.7) is recurrent.

Condition (4.7). Recall that we used notations $c = 2b/(1 - e^{-bt})\sigma^2, \theta = a + \sigma^2\eta$, and $u = ce^{-bT}/\xi$ in Section 3.2. For convenience, we consider c and u as functions of t, i.e.

$$c(t) = \frac{2b}{(1 - e^{-bt})\sigma^2}, \quad \text{and} \quad u(t) = \frac{c(t)e^{-bt}}{\xi}, \quad t \geq 0.$$

We have an explicit formula for $\mathbb{E}_\xi^{\mathbb{Q}}[1/\pi(X_{T-t})]$, which we have shown in Section 3.2:

$$\mathbb{E}_\xi^{\mathbb{Q}}\left[\frac{1}{\pi(X_{T-t})}\right] = \mathbb{E}_\xi^{\mathbb{Q}}[X_{T-t}^\eta] = c(T-t)^\eta e^{-u(T-t)}\frac{1}{\Gamma(\eta)}\int_0^1 e^{u(T-t)z} z^{2\theta/\sigma^2+1-\eta}(1-z)^{\eta-1}dz.$$

for each $0 \le t < T$. Since $e^{u(T-t)z}$ converges to 1 as $T \to \infty$ uniformly on $[0,1]$, we have

$$\lim_{T\to\infty} \mathbb{E}_{\xi}^{\mathbb{Q}}\left[\frac{1}{\pi(X_{T-t})}\right] = \left(\frac{2b}{\sigma^2}\right)^{\eta} \frac{\Gamma(2\theta/\sigma+2-\eta)}{\Gamma(2\theta/\sigma^2+2)}, \tag{4.10}$$

whence

$$\frac{\mathbb{E}_{X_t(\omega)}^{\mathbb{Q}^\pi}\left[\pi(X_{T-t})^{-1}\right]}{\mathbb{E}_{\xi}^{\mathbb{Q}^\pi}\left[\pi(X_T)^{-1}\right]} \longrightarrow 1 \text{ as } T \to \infty \tag{4.11}$$

for each $\omega \in \Omega$. For a given $t > 0$ fix $T_0 > t$. Then for all $T \ge T_0$,

$$\begin{aligned}
\mathbb{E}_{X_t}^{\mathbb{Q}^\pi}\left[\pi(X_{T-t})^{-1}\right] &\le c(T_0-t)^{\eta}\frac{1}{\Gamma(\eta)}\int_0^1 z^{2\theta/\sigma^2+1-\eta}(1-z)^{\eta-1}dz \\
&= \left(\frac{2b}{(1-e^{-b(T_0-t)})\sigma^2}\right)^{\eta}\frac{\Gamma(2\theta/\sigma^2+2-\eta)}{\Gamma(2\theta/\sigma^2+2)}.
\end{aligned}$$

Hence for such T_0 the ratio in (4.11) is bounded by some constant(depending on t) for all $T \ge T_0$. We conclude that L^1-convergence also holds.

Condition (4.2) *and* (2) These can be done by the same way as in the CIR model.

By Proposition 2.3.6, an invariant probability density of X here is given by

$$\psi(y) := \frac{(2b/\sigma^2)^{2\theta/\sigma^2+2}}{\Gamma(2\theta/\sigma^2+2)} y^{-\frac{2\theta}{\sigma^2}-3} e^{-2b/(\sigma^2 y)}.$$

As in Remark 4.4, it is easy to see that

$$\int_0^\infty y^\eta \psi(y)dy$$

coincides with (4.10) and this is the result of Theorem 2.3.7, but the function $1/\pi(x) = x^\eta$, $x > 0$ is not bounded.

Results

The only remaining part of the results of Theorem 4.2.4 we have to verify is (4.3). Put

$$L(x) = \left(\frac{2b}{(1-x^{-b})\xi\sigma^2}\right)^{\eta} \exp\left\{-\frac{2b}{(x^b-1)\xi\sigma^2}\right\}\frac{1}{\Gamma(\eta)} \cdot \int_0^1 \exp\left\{\frac{2b}{(x^b-1)\xi\sigma^2}z\right\}z^{2\theta/\sigma^2+1-\eta}(1-z)^{\eta-1}dz.$$

Then for any $a > 0$,

$$\lim_{x\to\infty} L(ax) = \left(\frac{2b}{\sigma^2}\right)^{\eta} \frac{\Gamma(2\theta/\sigma^2+2-\eta)}{\Gamma(2\theta/\sigma^2+2)}.$$

Hence, L is a slowly varying function, and satisfies $L(e^t) = e^{\lambda t}\mathcal{P}_t$.

4.4 Some other models

The recurrence condition and the condition (4.7) ensure the existence of the long bond and the long forward measure, and give the concrete form of the factorization of the long bond. The condition (2) in Theorem 4.2.4 also ensures that the valuation process of the zero-coupon bond is written as the discounted value of a slowly varying function at the long-term discount rate λ. Without these conditions the only result we can make sure is (4.4).

The CIR model and the 3/2 model in the previous section have well-known form of distribution. This allowed us to verify the results in Theorem 4.2.4 explicitly. A question arises whether Theorem 4.2.4 and 4.2.6 can be applied to many other models. In this section, we introduce some models that are not as famous as the CIR model and the 3/2 model. These models have no known closed-form expression of distribution, so the explicit calculation might be almost impossible. Verification for the condition (4.7) is still ongoing, so we only present the partial results.

We begin this section with an important remark that makes our attempts more reasonable.

Remark 4.5. Qin and Linetsky (2016) proved that there exists at most one recurrent eigenfunction among all the positive eigenfunctions of pricing operator. Thus, if our approach is finding a recurrent eigenmeasure such that the condition (4.7) holds, then the recurrent eigenfunction for each model, if it exists, is the only candidate for the positive eigenfunction satisfying the conditions in Theorem 4.2.6.

4.4.1 Square diffusion model

Consider the SDE

$$dX_t = (b - aX_t)X_t^2 dt + \sigma X_t^2 dB_t, \quad X_0 = \xi \tag{4.12}$$

where $a, b, \sigma > 0$ and $\xi > 0$. The SDE (4.12) has a unique strong solution, and the solution takes values in $(0, \infty)$. The proof is given in Appendix B.1.

The equation (2.3) for the square diffusion model with killing rate $r(x) = x$ becomes

$$\frac{1}{2}\sigma^2 x^4 \pi''(x) + (b - ax)x^2 \pi'(x) - x\,\pi(x) = -\lambda \pi(x).$$

One has an eigenpair

$$(\lambda, \pi(x)) = \left(\frac{1}{\eta}\left(b - \frac{\sigma^2}{2\eta}\right), e^{1/\eta x}\right), \quad x > 0, \tag{4.13}$$

where $\eta = a + \sigma^2$.

A process

$$M_t := \exp\left\{\lambda t - \int_0^t X_s\, ds + \frac{1}{\eta}\left(\frac{1}{X_t} - \frac{1}{\xi}\right)\right\}, \ t \geq 0 \tag{4.14}$$

is then a local martingale. We shall make use of Corollary 2.2.5 to show that the process (4.14) is a martingale. Consider a weak solution of the SDE

$$dX_t = \left((b - aX_t)X_t^2 + \sigma^2 X_t^4 \frac{\pi'(X_t)}{\pi(X_t)}\right) dt + \sigma X_t^2 dB_t$$

$$= \left(b - \frac{\sigma^2}{a+\sigma^2} - aX_t\right) X_t^2 dt + \sigma X_t^2 dB_t$$

Using Proposition 2.3.4, one can see that X is non-explosive in $(0, \infty)$. Therefore, (4.14) is a true martingale.

Under the corresponding eigenmeasure $\mathbb{Q}^\pi$, the process

$$B_t^{\mathbb{Q}^\pi} = B_t + \frac{\sigma}{a+\sigma^2} t,\ t \geq 0$$

is a Brownian motion, and (4.12) becomes

$$dX_t = (\beta - aX_t) X_t^2 dt + \sigma X_t^2 dB_t^{\mathbb{Q}^\pi},$$

where $\beta = b - \sigma^2/(a+\sigma^2)$, and the solution X is recurrent in $(0, \infty)$ under $\mathbb{Q}^\pi$, provided that $\beta > 0$. Thus, if $\beta > 0$, then (4.13) is admssible and recurrent, and

$$\mathcal{P}_t = \mathbb{E}_\xi^{\mathbb{P}} \left[e^{-\int_0^t X_s\, ds} \right] = e^{-\lambda t - \eta\xi}\, \mathbb{E}_\xi^{\mathbb{Q}^\pi} \left[e^{-1/(\eta X_t)} \right].$$

We observe whether X satisfies the condition (4.7). One approach in progress is as follows:

The speed density of the square process under $\mathbb{Q}^\pi$ is given by

$$m(x) = Cx^{-2a/\sigma^2 - 4} e^{-2\beta/(\sigma^2 x)},\ C > 0.$$

By Proposition 2.3.6, an invariant probability density of X is given by

$$\psi(y) := \frac{(2\beta/\sigma^2)^{2a/\sigma^2+3}}{\Gamma(2a/\sigma^2+3)} y^{-\frac{2a}{\sigma^2}-4} e^{-2\beta/(\sigma^2 y)}.$$

The integral of $1/\pi$ under the invariant measure is then

$$\int_0^\infty e^{-1/(\eta y)} \psi(y) dy = \left(1 + \frac{\sigma^2}{2\beta\eta}\right)^{-2a/\sigma^2 - 3} \frac{\Gamma(2\beta/\sigma^2 + \eta^{-1} + 3)}{\Gamma(2\beta/\sigma^2 + 3)}. \quad (4.15)$$

We note that $1/\pi(x) = e^{-1/(\eta x)}$, $x > 0$ is bounded. Once the density assumption in Theorem 2.3.7 is satisfied for X under $\mathbb{Q}^\pi$, by Theorem 2.3.7

$\mathbb{E}_\xi^{\mathbb{Q}^\pi}[e^{1/(\eta X_T)}]$ converges to the value (4.21), and eventually the condition (4.7) will be ensured.

Put $Y_t = 1/(\sigma X_t)$, $t \geq 0$. Applying Itô formula one can see that the process Y is a solution of the SDE

$$dY_t = \left(\left(\frac{a}{\sigma} + \sigma \right) \frac{1}{Y_t} - \beta \right) dt - dB_t^{\mathbb{Q}^\pi}. \tag{4.16}$$

Define

$$\begin{aligned} Z_t &= \exp\left\{ -\beta B_t - \frac{1}{2}\beta^2 t \right\}, \\ \widetilde{B}_t &= -B_t^{\mathbb{Q}^\pi} - \beta t. \end{aligned}$$

Then a new measure $\widetilde{\mathbb{Q}}$ is defined on each $\mathcal{F}_t$, $t \geq 0$ as $\widetilde{\mathbb{Q}}|_{\mathcal{F}_t} = Z_t \mathbb{Q}^\pi|_{\mathcal{F}_t}$, and $\widetilde{B}$ is a Brownian motion under $\widetilde{\mathbb{Q}}$ by the Girsanov theorem. Under $\widetilde{\mathbb{Q}}$, (4.16) is rewritten as a Bessel process of dimension $a/\sigma + \sigma$:

$$dY_t = \left(\frac{a}{\sigma} + \sigma \right) \frac{1}{Y_t} dt + d\widetilde{B}_t.$$

Hence, there is a strictly positive and continuous function $p \in C((0,\infty) \times I^2)$ such that

$$\widetilde{\mathbb{Q}}_\xi (Y_t \in dy) = p(t, \xi, y) dy$$

for all $t \geq 0$ and $\xi \in I$(Jeanblac et al., 2009). On the other hand, since $\widetilde{\mathbb{Q}}$ and $\mathbb{Q}^\pi$ are locally equivalent, their transition measures are equivalent. Thus, there exists a strictly positive function f on $(0,\infty) \times I^2$ such that

$$\mathbb{Q}_\xi^\pi (Y_t \in dy) = f(t, \xi, y) \widetilde{\mathbb{Q}}_\xi (Y_t \in dy)$$

for all $t \geq 0$ and $\xi \in I$. Therefore, the transition probability of Y under $\mathbb{Q}^\pi$ has the strictly positive transition density, so does that of X.

It remains to show that the transition density of X is continuous. Intuitively it seems true, but it is not clear. This problem would be one of the future works.

4.4.2 The inverse GARCH model

Consider the SDE

$$dX_t = (b - aX_t)X_t dt + \sigma X_t dB_t, \quad X_0 = \xi \tag{4.17}$$

where $a, b, \sigma > 0$ and $\xi > 0$. To show the existence and uniquess of the strong solution of (4.17), observe its reciprocal process $Y_t := 1/X_t$. It follows the GARCH linear SDE

$$dY_t = (a - (b - \sigma^2)Y_t)dt - \sigma Y_t dB_t, \quad Y_0 = \frac{1}{\xi} \tag{4.18}$$

with constant coefficients. Hence, the SDE has the unique strong solution(Karatzas and Shreve, 1998, Chapter 5, Theorem 2.9.). Furthermore, (2.3.4) guarantees that under the condition $2b > \sigma^2$, Y always takes values in $(0, \infty)$ with $Y_0 = 1/\xi > 0$. Thus, the SDE (4.17) has the unique strong solution and X stays positive.

Next we look for a recurrent eigenpair. The equation (2.3) for the inverse GARCH model with killing rate $r(x) = x$ is

$$\frac{1}{2}\sigma^2 x^2 \pi''(x) + (b - ax)x\,\pi'(x) - x\,\pi(x) = -\lambda\pi(x).$$

One eigenpair can be obtained by direct calculuation, that

$$(\lambda, \pi(x)) = \left(\left(b - \frac{\sigma^2}{2}\right)\frac{1}{a} - \frac{\sigma^2}{2a^2}, x^{-1/a}\right). \tag{4.19}$$

Denote the corresponding eigenmeasure of the eigenpair as $\mathbb{Q}^\pi$.

A process

$$M_t := e^{\lambda t - \int_0^t X_s\,ds}\left(\frac{X_t}{\xi}\right)^{-1/a}, \, t \geq 0 \tag{4.20}$$

is a local martingale. In fact, it is a true martingale. The proof is given in Appendix C.2. Under the corresponding eigenmeasure $\mathbb{Q}^\pi$, the process

$$B_t^{\mathbb{Q}^\pi} = B_t + \frac{\sigma}{a}t, \, t \geq 0$$

is a Brownian motion, and (4.17) becomes

$$dX_t = (\beta - aX_t)X_t dt + \sigma X_t dB_t^{\mathbb{Q}^\pi}, \quad X_0 = \xi,$$

where $\beta = b - \sigma^2/a$. Provided that $2b > (2/a+1)\sigma^2$, X is recurrent by Proposition 2.3.4, and the eigenpair (4.19) is admissible and recurrent. The valuation process of the zero-coupon bond is given by

$$\mathcal{P}_t = \mathbb{E}_\xi^{\mathbb{P}}\left[e^{-\int_0^t X_s\,ds}\right] = e^{-\lambda t}\xi^{-1/a}\,\mathbb{E}_\xi^{\mathbb{Q}^\pi}\left[X_t^{1/a}\right].$$

The speed density of the inverse GARCH process under $\mathbb{Q}^\pi$ is given by

$$m(x) = Cx^{2\beta/\sigma^2-2}e^{-(2a/\sigma^2)x},\ C > 0.$$

By Proposition 2.3.6,

$$\psi(y) := \frac{(2a/\sigma^2)^{2\beta/\sigma^2-1}}{\Gamma(2\beta/\sigma^2-1)}y^{2\beta/\sigma^2-2}e^{-(2a/\sigma^2)y}$$

is an invariant probability density of X. This is the density of a Gamma distribution $\Gamma(2\beta/\sigma^2-1, \sigma^2/2a)$. The integral of $1/\pi$ under the invariant measure is then

$$\int_0^\infty y^{1/a}\psi(y)dy = \left(\frac{\sigma^2}{2a}\right)^{1/a}\frac{\Gamma(2\beta/\sigma^2+a^{-1}-1)}{\Gamma(2\beta/\sigma^2-1)}. \tag{4.21}$$

Consider the GARCH linear model (4.18). Since $b(x) = a - (b-\sigma^2)x$ has linear growth and $\sigma(x) = \sigma x$ is Lipschitz continuous, the transition density exists by Fournier and Printems, 2010, Theorem 2.1. Thus, X also has the transition density.

The continuity of the density function is not clear, though. Moreover, the function $1/\pi(x) = x^{1/a}$, $x > 0$ is not bounded, so we cannot apply Theorem 2.3.7. Hence whether the process X satisfies the condition (4.7) is not clear yet.

4.4.3 Quadratic model

Consider the SDE

$$dX_t = (b - aX_t^2)dt + \sigma X_t dB_t, \quad X_0 = \xi \tag{4.22}$$

where $a, b, \sigma > 0$ and $\xi > 0$. The SDE (4.22) has a unique strong solution, and the solution takes values in $(0, \infty)$. The proof is given in Appendix B.2.

Let $r(x) = x^2$ be the killing rate. Then the equation (2.3) for the quadratic model becomes

$$\frac{1}{2}\sigma^2 x^2 \pi''(x) + (b - ax^2)\pi'(x) - x^2\pi(x) = -\lambda\pi(x).$$

By direct calculation one has an eigenpair

$$(\lambda, \pi(x)) = (b\eta, e^{-\eta x}), \quad x > 0 \tag{4.23}$$

where

$$\eta = \frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2}.$$

A process

$$M_t := \exp\left\{\lambda t - \int_0^t X_s^2 ds - \eta X_t + \eta\xi\right\},\ t \geq 0$$

is then a local martingale, as we saw in section 2.2. Moreover, since

$$\mathbb{E}_\xi^{\mathbb{P}}\left[\sup_{0\leq s\leq t} |M_s|\right] \leq \mathbb{E}_\xi^{\mathbb{P}}\left[e^{b\eta t + \eta\xi}\right] < \infty \quad \forall t \geq 0,$$

by Proposition 2.2.2 M is a true martingale. Under the corresponding eigenmeasure $\mathbb{Q}^\pi$, the process

$$B_t^{\mathbb{Q}^\pi} = B_t + \sigma\eta \int_0^t X_s\, ds,\ t \geq 0$$

is a Brownian motion, and (4.22) becomes

$$dX_t = (b - \alpha X_t^2)dt + \sigma X_t dB_t^{\mathbb{Q}^\pi}. \tag{4.24}$$

where $\alpha = \sqrt{a^2 + 2\sigma^2}$. Using Proposition 2.3.4, one can see that X is recurrent in $(0, \infty)$ under $\mathbb{Q}^\pi$. Thus, (4.23) is admissible and recurrent. The valuation process of the zero-coupon bond is given by

$$\mathcal{P}_t = \mathbb{E}_\xi^{\mathbb{P}}\left[e^{-\int_0^t X_s^2 ds}\right] = e^{-\lambda t - \eta\xi}\,\mathbb{E}_\xi^{\mathbb{Q}^\pi}[e^{\eta X_t}].$$

The speed density of the Quadratic process under $\mathbb{Q}^\pi$ is given by

$$m(x) = \frac{C}{x^2}\exp\left\{-\frac{2b}{\sigma^2 x} - \frac{2\alpha}{\sigma^2}x\right\},\ C > 0.$$

Choose C satisfying $\int_I m(x)dx = 1$. Then by Proposition 2.3.6 m is the invariant probability density of X. Since $\eta = (\alpha - a)/\sigma^2 < 2\alpha/\sigma^2$, the integral

$$\int_0^\infty e^{\eta y} m(y)dy = C\int_0^\infty \frac{1}{y^2}\exp\left\{-\frac{2b}{\sigma^2 y} - \left(\frac{2\alpha}{\sigma^2} - \eta\right)y\right\}dy$$

exists and is finite.

Define

$$\begin{aligned} Z_t &= \exp\left\{\int_0^t \frac{\alpha}{\sigma}X_s\,dB_s - \frac{1}{2}\int_0^t \frac{\alpha^2}{\sigma^2}X_s^2 ds\right\}, \\ \widetilde{B}_t &= B_t^{\mathbb{Q}^\pi} - \int_0^t \frac{\alpha}{\sigma}X_s ds. \end{aligned}$$

We claim that the process Z is a true martingale. To this end, we shall show that Z satisfies the Novikov condition(Karatzas and Shreve, 1998, Corollary 5.13). Thanks to Jensen's inequality, Fubini's theorem, and the estimate (B.4) in Appendix B

$$\mathbb{E}_\xi^{\mathbb{Q}^\pi}[X_t^2] \leq (1 + \xi^2)e^{Ct} - 1, \quad \forall t \geq 0$$

for some $C > 0$, the Novikov condition holds:

$$\begin{aligned}\mathbb{E}_\xi^{\mathbb{Q}^\pi 0}\left[\exp\left\{\frac{1}{2}\int_0^t \frac{\alpha^2}{\sigma^2} X_s^2\, ds\right\}\right] &\leq \exp\left\{\frac{\alpha^2}{2\sigma^2}\int_0^t \mathbb{E}_\xi^{\mathbb{Q}^\pi}[X_s^2] ds\right\}\\ &\leq \exp\left\{\frac{\alpha^2}{2\sigma^2}\int_0^t (1+\xi^2)e^{Cs} - 1 ds\right\}\\ &< \infty\end{aligned}$$

for all $t \geq 0$. Thus, a new measure $\widetilde{\mathbb{Q}}$ is defined on each $\mathcal{F}_t$, $t \geq 0$ as $\widetilde{\mathbb{Q}}|_{\mathcal{F}_t} = Z_t \mathbb{Q}^\pi|_{\mathcal{F}_t}$, and $\widetilde{B}$ is a Brownian motion under $\widetilde{\mathbb{Q}}$ by Girsanov's theorem. Under $\widetilde{\mathbb{Q}}$, the SDE (4.24) is given by

$$dX_t = b\, dt + \sigma X_t dB_t^{\widetilde{\mathbb{Q}}}.$$

Since $b(x) = b$ is a constant and $\sigma(x) = \sigma x$ is Lipschitz continuous, the transition density exists by Theorem 2.1 in Fournier and Printems (2010). The remaining part can be demonstrated by employing the same argument as in the case of the square diffusion model.

The continuity of the density function here is still unclear, and the function $1/\pi(x) = e^{\eta x}$, $x > 0$ is not bounded. Thus, we cannot apply Theorem 2.3.7, and whether the process X satisfies the condition (4.7) in this case is also unknown yet.

Chapter 5

Conclusion

This paper introduced the martingale extraction method of Leung and Park (2017) originated from Hansen and Scheinkman (2009), and exploited the method to factorize the stochastic discount factor in some models under Markovian assumption. We also developed some criteria for a local martingale to be a true martingale. On the basis of the works of Qin and Linetsky (2017), we found explicit forms of the factorization in the CIR model and in the 3/2 model with detailed verifications. The unsolved problems appear in analysis of the models presented in Section 4.4 are subject to future works.

The common problem appeard in all the three models in Section 4.4 was to prove continuity of the transition density function. To solve this problem, we have to develop some methods useful for proving continuity of the density, or have to relax the continuity assumption. The boundedness assumption imposed on an integrand in Theorem 2.3.7 also has to be weakened, so that one can apply the ergodic theorem for larger class of functions. Investigating other approaches such as spectral theory would be one option for solving the problems.

The method of martingale extraction turns out to be a useful tool in the field of *long-term* analysis, as we saw, for example, in Hansen and Scheinkman (2009) and Leung and Park (2017). Besides, this method would also be useful for various research areas related to quantitative finance, because one may turn the problem of analyzing a path-dependent object into analyzing a path-independent one.

Appendix A

Noncentral χ^2-distribution and related processes

In this chapter, we first introduce the notion of a noncentral χ^2-distribution, and then investigate the distribution of a CIR process and a 3/2 process, which are closely related to the noncentral χ^2-distribution.

A.1 Noncentral χ^2-distributions

Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim N(\mu_i, 1)$ for each $i = 1, 2, \ldots, n$. Put $Y = \sum_{i=1}^{n} X_i^2$ and $\theta = \sum_{i=1}^{n} \mu_i^2$. Then Y is said to have a noncentral χ^2-distribution with two parameters: n degrees of freedom and θ, which is called the noncentral parameter. We use the notation $\chi^2(n, \theta)$ to denote such a random variable. The moment generating function of Y is easily derived from the moment generating function of a normal distribution,

$$M_Y(t) = \mathbb{E}[e^{tY}] = \frac{1}{(1-2t)^{n/2}} \exp\left\{\frac{\theta t}{\sigma^2(1-2t)}\right\}, \quad t < \frac{1}{2}. \tag{A.1}$$

Also, the distribution function and the density function of Y are given by

$$F_Y(y;n,\theta) = e^{-\theta/2}\sum_{m=1}^{\infty}\frac{(\theta/2)^m/m!}{2^{n/2+m}\Gamma(n/2+m)}\int_0^y z^{n/2+m-1}e^{-z/2}dz$$

for $y > 0$ and

$$f_Y(y;n,\theta) = \frac{1}{2}\left(\frac{y}{\theta}\right)^{(n-2)/4} e^{-(\theta+y)/2} I_{(n-2)/2}(\sqrt{\theta y})\mathbb{I}_{\{y>0\}} \tag{A.2}$$

(Patnaik, 1949), where

$$I_q(y) = \left(\frac{1}{2}y\right)^q \sum_{m=1}^{\infty}\frac{(y^2/4)^m}{m!\Gamma(q+m+1)} \tag{A.3}$$

is a modified Bessel function of the first kind of order q.

In the above construction of $\chi^2(n,\theta)$, n is a positive integer. However, the distribution $\chi^2(\nu,\theta)$ for a positive real number ν also makes sense(Johnson et al., 1995, page 436). Indeed, for any $\nu > 0$ it is clear that

$$\lim_{y\to 0+} F_Y(y;\nu,\theta) = 0, \ \lim_{y\to\infty} F_Y(y;\nu,\theta) = 1, \text{ and } F_Y \text{ is increasing and continuous.}$$

Definition A.1.1 (Noncentral χ^2-distribution)**.** A random variable Y that has the distribution function

$$F_Y(y;\nu,\theta) = e^{-\theta/2}\sum_{m=1}^{\infty}\frac{(\theta/2)^m/m!}{2^{\nu/2+m}\Gamma(\nu/2+m)}\int_0^y z^{\nu/2+m-1}e^{-z/2}dz, \quad \nu > 0, \theta > 0$$

is said to have a noncentral χ^2-distribution with ν degrees of freedom and noncentrality parameter θ, and is denoted as $Y \sim \chi^2(\nu,\theta)$.

Moreover, differentiability of F_Y for arbitrary ν ensures that the density function exists and is of the form (A.2). Finally, the moment generating function of $Y \sim \chi^2(\nu,\theta)$ for arbitrary $\nu > 0$ is also of the form (A.1), which can be obtained by direct calculation

$$M_Y(t) = \int_0^{\infty} e^{ty}\frac{1}{2}\left(\frac{y}{\theta}\right)^{(\nu-2)/4} e^{-(\theta+y)/2}\left(\frac{1}{2}y\right)^{(\nu-2)/2}\sum_{m=1}^{\infty}\frac{(y^2/4)^m}{m!\Gamma(\frac{\nu-2}{2}+m+1)}dy$$

(Put $z = (1-2t)y$ and $\lambda = \theta/(1-2t)$)

$$= \frac{1}{(1-2t)^{\nu/2}} \exp\left\{\frac{\theta t}{\sigma^2(1-2t)}\right\}$$
$$\cdot \int_0^\infty \frac{1}{2}\left(\frac{z}{\lambda}\right)^{(\nu-2)/4} e^{-(\lambda+z)/2}\left(\frac{1}{2}z\right)^{(\nu-2)/2}\sum_{m=1}^{\infty}\frac{(z^2/4)^m}{m!\Gamma(\frac{\nu-2}{2}+m+1)}dz$$

$$= \frac{1}{(1-2t)^{\nu/2}} \exp\left\{\frac{\theta t}{\sigma^2(1-2t)}\right\}\int_0^\infty f_Z(z;\nu,\lambda)dz \quad \left(Z \sim \chi^2(\nu,\lambda)\right)$$

$$= \frac{1}{(1-2t)^{\nu/2}} \exp\left\{\frac{\theta t}{\sigma^2(1-2t)}\right\}, \quad t < \frac{1}{2}. \tag{A.4}$$

By (A.4) and induction argument, the k-th moment can be written in the form

$$\mathbb{E}[Y^k] = \sum_{\substack{i+j\le k\\ i,j\ge 0}} a_{ij}\nu^i\theta^j, \quad k \in \mathbb{N}. \tag{A.5}$$

where $(a_{ij})_{i,j\ge 0}$ are constants which are independent of parameters ν and θ.

A.2 Distribution of a CIR model

Consider a CIR process

$$dX_t = (b - aX_t)dt + \sigma\sqrt{X_t}dB_t, \quad X_0 = \xi. \tag{A.6}$$

According to Cox et al., 1985, Page 391, the conditional density function of the CIR process (A.6) is given by

$$f(X_t; a, b, \sigma | X_0 = \xi) = ce^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}),$$

where

$$c = \frac{2a}{(1-e^{-at})\sigma^2},\ q = \frac{2b}{\sigma^2} - 1,\ u = ce^{at}\xi,\ v = cX_t,$$

and I_q is a modified Bessel function of the first kind of order q (A.3). Then X_t can be written as

$$X_t = \frac{Y}{2c}, \quad Y \sim \chi^2\left(\frac{4b}{\sigma^2}, 2ce^{-at}\xi\right).$$

Put $\nu = 4b/\sigma^2$ and $\theta = 2ce^{-at}\xi$. Using the moment generating function of the noncentral χ^2-distribution (A.1), we can find the moment generating function of X_t,

$$M_{X_t}(s) = \mathbb{E}\left[e^{sX_t}\right] = \mathbb{E}\left[e^{\frac{s}{2c}Y}\right] = \left(\frac{1}{1-s/c}\right)^{2b/\sigma^2} \exp\left\{\frac{2ce^{-at}\xi s}{\sigma^2(1-s/c)}\right\}, \quad s < c. \tag{A.7}$$

Finally, by (A.5) the k-th conditional moment is written in the form

$$\mathbb{E}[X_t^k|X_0 = \xi] = \frac{1}{(2c)^k}\mathbb{E}[Y^k] = \frac{1}{(2c)^k} \sum_{\substack{i+j\le k \\ i,j\ge 0}} a_{ij} \left(\frac{4b}{\sigma^2}\right)^i (2ce^{-at}\xi)^j. \quad k \in \mathbb{N},$$

A.3 Distribution of 3/2 model

We investigate the distribution of a 3/2 process

$$dX_t = (b - aX_t)X_t\,dt + \sigma X_t^{3/2}\,dB_t, \;\; X_0 = \xi,$$

which is closely related to a CIR process as we saw in subsection 3.2. Here we follow the exposition of Ahn and Gao (1999).

We recall that the reciprocal of X follows a CIR process (3.6). Since we already investigated the conditional density function of the CIR process in Appendix A.2, now it is easy to find the density function of (3.5). It is given by

$$f(X_t; a, b, \sigma|X_0 = \xi) = c^{-1}e^{-u-v}v^{p/2+2}u^{-p/2}I_p(2\sqrt{uv}) \tag{A.8}$$

where

$$c = \frac{2b}{\sigma^2(1-e^{-bt})}, \; u = \frac{c}{\xi}e^{-bt}, \; v = \frac{c}{X_t}, \; p = \frac{2a}{\sigma^2} + 1,$$

and I_p is a modified Bessel function of the first kind of order p (A.3). Straightforward calculation of the ν-th conditional moment for X_t using the density (A.8) gives

$$\mathbb{E}[X_t^\nu | X_0 = \xi] = c^\nu e^{-u} \frac{1}{\Gamma(\nu)} \int_0^1 e^{uz} z^{p-\nu} (1-z)^{\nu-1} dz, \quad 0 < \nu < p+1. \tag{A.9}$$

In fact, ν need not be a postive integer, but it may be a postive real number.

Appendix B

Proofs for existence and uniqueness problems

In this chapter, we prove the existence and uniqueness of a strong solution of the square diffusion model and the quadratic model introduced in Section 4.4.

B.1 The square diffusion model

Consider a SDE

$$dY_t = \left(2a + 3\sigma^2 - 2b\sqrt{Y_t}\right) dt - 2\sigma\sqrt{Y_t} dB_t, \quad Y_0 = \zeta \tag{B.1}$$

for $a, b, \sigma > 0$ and $\zeta > 0$ is deterministic. Since the SDE (B.1) satisfies a condition of linear growth, the weak existence holds. It is straightforward to check by Proposition 2.3.4 that a weak solution of the SDE (B.1) takes values in $(0, \infty)$. Hence, a process $X_t := 1/\sqrt{Y_t}$, $t \geq 0$ is well-defined.

Put $f(y) = 1/\sqrt{y}$, $y > 0$. Applying Itô formula to $X_t = f(Y_t)$, $t \geq 0$ yields (4.12). Thus, the SDE (4.12) has a weak solution. Since $b(x) = (b - ax)x^2$ and $\sigma(x) = \sigma x^2$ in (4.12) are locally Lipschitz continuous, the pathwise uniqueness holds for (4.12)(Ikeda and Watanabe, 1989, Chapter

4, Theorem 3.1). Therefore, we conclude that the SDE (4.12) has a unique strong solution(Ikeda and Watanabe, 1989, Chapter 4, Theorem 1.1).

B.2 The quadratic model

The following lemma is a slight modification of Ikeda and Watanabe, 1989, Chapter 4, Theorem 2.4.

Lemma B.2.1. *Consider the SDE* (2.1)*. If*

1. *b and σ are continuous,*
2. *$\exists C > 0$ such that $x\, b(x) \leq C(1+x^2)\, \forall x \in \mathbb{R}$,*
3. *$\exists C > 0$ such that $|\sigma(x)|^2 \leq C(1+x^2)\, \forall x \in \mathbb{R}$,*

then the SDE has a non-explosive(global) weak solution.

Proof. By Ikeda and Watanabe, 1989, Chapter 4, Theorem 2.3, there is a local weak solution $(X, W), (\widetilde{\Omega}, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ up to explosion time. Hence, if we put $\sigma_n = \inf\{t > 0 : |X_t| \geq n\}$ for $n \in \mathbb{N}$ and $f(x) = x^2$, and apply Itô formula, we have

$$X^2_{t\wedge\sigma_n} = \xi^2 + \int_0^{t\wedge\sigma_n} (2X_s b(X_s) + \sigma^2(X_s))ds + \int_0^{t\wedge\sigma_n} 2X_s\sigma(X_s)dW_s.$$

In addition, by assumptions on b and σ we have for some constant $C > 0$ that

$$X^2_{t\wedge\sigma_n} \leq \xi^2 + C\int_0^{t\wedge\sigma_n} (1 + X_s^2)ds + \int_0^{t\wedge\sigma_n} 2X_s\sigma(X_s)dW_s. \tag{B.2}$$

Since $\int_0^{\cdot\wedge\sigma_n} 2X_s\sigma(X_s)dW_s$ is a martingale, we have by (B.2) and Itô isometry

$$\widetilde{\mathbb{E}}[X^2_{t\wedge\sigma_n}] \leq \xi^2 + C\,\widetilde{\mathbb{E}}\left[\int_0^{t\wedge\sigma_n} (1 + X_s^2)ds\right]$$

$$
\begin{aligned}
&= \xi^2 + C\,\widetilde{\mathbb{E}}\left[\int_0^t (1 + X_s^2)\mathbb{I}_{\{s \le \sigma_n\}} ds\right] \\
&\le \xi^2 + C\int_0^t (1 + \widetilde{\mathbb{E}}[X_{s\wedge\sigma_n}^2])ds, \qquad \text{(B.3)}
\end{aligned}
$$

where $\widetilde{\mathbb{E}}$ is the expectation with respect to $\widetilde{\mathbb{P}}$. Applying Gronwall's inequality to (B.3) yields

$$\widetilde{\mathbb{E}}[X_{t\wedge\sigma_n}^2] \le (1 + \xi^2)e^{Ct} - 1$$

Letting $n \to \infty$, by monotone convergence theorem we conclude that

$$\widetilde{\mathbb{E}}[X_t^2] \le (1 + \xi^2)e^{Ct} - 1 \qquad \text{(B.4)}$$

for all $t \ge 0$. □

Unfortunately, in the quadratic model (4.22) $b(x) = b - ax^2$ does not satisfy the condition 2 in Lemma B.2.1. Hence we take a detour to apply Lemma B.2.1.

Consider a SDE

$$d\tilde{X}_t = \mu(\tilde{X}_t)dt + \sigma\tilde{X}_t dB_t, \quad \tilde{X}_0 = \xi > 0, \qquad \text{(B.5)}$$

where $\mu : \mathbb{R} \to \mathbb{R}$ is defined as

$$\mu(x) = \begin{cases} b - ax^2 & \text{for } x \ge 0 \\ b & \text{for } x < 0 \end{cases}.$$

Clearly μ is continuous and there exists a constant $C > 0$ such that $x\mu(x) \le C(1 + x^2)$ for all $x \in \mathbb{R}$. Thus, (B.5) has a weak solution. Furthermore, applying Proposition 2.3.4 shows that the weak solution takes values in $(0, \infty)$. Therefore, for given strictly positive deterministic initial condition the SDE (4.22) and (B.5) have the same weak solutions.

Finally, since $b(x)$ and $\sigma(x)$ in (4.22) are locally Lipschitz continuous, the pathwise uniqueness holds for (4.22)(Ikeda and Watanabe, 1989, Chapter 4, Theorem 3.1). Therefore, we conclude that the SDE (4.22) has a unique strong solution(Ikeda and Watanabe, 1989, Chapter 4, Theorem 1.1).

Appendix C

Proofs for martingales

In this chapter, we confirm that local martingales (3.8) and (4.20) are true martingales.

C.1 The 3/2 model

In differential form, (3.8) is written in the form

$$dM_t = -\sigma\eta\xi^{\eta}e^{\lambda t - \int_0^t X_s\,ds}X_t^{-\eta+\frac{1}{2}}dB_t.$$

We claim that

$$\mathbb{E}_\xi^{\mathbb{P}}\left[\int_0^t X_s^{1-2\eta}ds\right] < \infty, \quad t \geq 0. \tag{C.1}$$

Once the claim is proven, then

$$\mathbb{E}_\xi^{\mathbb{P}}\left[\int_0^t e^{2\lambda s - 2\int_0^s X_u\,du}X_s^{1-2\eta}ds\right] \leq e^{\lambda t}\mathbb{E}_\xi^{\mathbb{P}}\left[\int_0^t X_s^{1-2\eta}ds\right] < \infty,$$

and the proof ends.

If $1-2\eta < 0$, by Funibi's theorem and Jensen's inequality

$$\mathbb{E}_\xi^{\mathbb{P}}\left[\int_0^t X_s^{1-2\eta}ds\right] \leq \int_0^t \mathbb{E}_\xi^{\mathbb{P}}\left[\left(\frac{1}{X_s}\right)^{2\lceil\eta\rceil-1}\right]^{\frac{2\eta-1}{2\lceil\eta\rceil-1}} ds.$$

We note that $1/X_s$ follows a CIR process and by (A.2) that $\mathbb{E}_\xi^{\mathbb{P}}\left[(1/X_s)^{2\lceil\eta\rceil-1}\right]$ is a continuous function of s in $[0,t]$(It has no singularities at all). Thus, (C.1) holds.

In the case of $1-2\eta>0$, we use the conditional moment (A.9) of a 3/2 process. By Funibi's theorem and Jensen's inequality

$$\mathbb{E}_\xi^{\mathbb{P}}\left[\int_0^t X_s^{1-2\eta}ds\right] \le \int_0^t \mathbb{E}_\xi^{\mathbb{P}}\left[X_s\right]^{1-2\eta} ds, \tag{C.2}$$

and we shall show that the right hand side of (C.2) is finite, by arguing that $s \mapsto \mathbb{E}[X_s]^{1-2\eta}$ is dominated by an integrable function on $[0,t]$. The formula (A.9) for $\nu=1$ is written as

$$\mathbb{E}[X_t|X_0=\xi] = ce^{-u}\int_0^1 e^{uz}z^{p-1}dz,$$

where

$$c=\frac{2b}{\sigma^2(1-e^{-bt})},\ u=\frac{c}{\xi}e^{-bt},\ \text{and } p=\frac{2a}{\sigma^2}+1,$$

Using integration by parts formula,

$$\begin{aligned}
\mathbb{E}[X_s|X_0=\xi] &= ce^{-u}\int_0^1 e^{uz}z^{p-1}dz \\
&= \xi e^{bs}e^{-u}\int_0^1 ue^{uz}z^{p-1}dz \qquad \left(p-1=2a/\sigma^2>0\right) \\
&= \xi e^{bs}e^{-u}\left(e^u-(p-1)\int_0^1 e^{uz}z^{p-2}dz\right) \\
&\le \xi e^{bs}e^{-u}\left(e^u+e^u(p-1)\int_0^1 z^{p-2}dz\right) \\
&= 2\xi e^{bs}.
\end{aligned}$$

Therefore,

$$\int_0^t \mathbb{E}_\xi^{\mathbb{P}}\left[X_s\right]^{1-2\eta} ds \le 2\xi\int_0^t e^{bs(1-2\eta)}ds < \infty.$$

C.2 The inverse GARCH model

In differential form, (4.20) is written in the form

$$dM_t = -e^{\lambda t - \int_0^t X_s\, ds} \xi^{1/a} \frac{\sigma}{a} X_t^{-1/a} dB_t.$$

Hence, to show that (4.20) is a true martingale, it suffices to show that for each $t \geq 0$

$$\mathbb{E}_\xi^{\mathbb{P}} \left[\int_0^t X_s^{-2/a} ds \right] < \infty.$$

We have already seen that the process $Y_t = 1/X_t$ follows the linear SDE (4.18). The explicit solution of a linear SDE is well-known. In particular,

$$\frac{1}{X_t} = Y_t = \frac{1}{\xi} e^{-(b-\sigma^2/2)t - \sigma B_t} + a \int_0^t e^{-(b-\sigma^2/2)(t-s) - \sigma(B_t - B_s)} ds$$

(Klebaner, 2012, p132). Since $2b > \sigma^2$ is a given condition, we have

$$\begin{aligned} X_t^{-2/a} &\leq \left(\frac{1}{\xi} e^{-\sigma B_t} + a \int_0^t e^{-\sigma(B_t - B_s)} ds \right)^{2/a} \\ &\leq \max\left\{1, 2^{2/a-1}\right\} \left(\left(\frac{1}{\xi}\right)^{2/a} e^{-(2\sigma/a)B_t} + a^{2/a} \left(\int_0^t e^{-\sigma(B_t-B_s)} ds \right)^{2/a} \right) \end{aligned} \tag{C.3}$$

where the second inequality comes from two elementary inequalities, that for $\alpha, \beta > 0$,

$$(\alpha+\beta)^p \leq \alpha^p + \beta^p \text{ for } 0 < p \leq 1, \quad \text{and} \quad (\alpha+\beta)^p \leq 2^{p-1}(\alpha^p + \beta^p) \text{ for } p > 1.$$

From (C.3), it sufficies to show that

1. $t \mapsto \mathbb{E}^{\mathbb{P}}[e^{-(2\sigma/a)B_t}]$ and
2. $t \mapsto \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^t e^{-\sigma(B_t - B_s)} ds \right)^{2/a} \right]$ are locally integrable.

The first one is trivial. For the second one, suppose $2/a > 1$ first. By Jensen's inequality and Fubini's theorem, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^t e^{-\sigma(B_t-B_s)}ds\right)^{2/a}\right] &\leq \mathbb{E}^{\mathbb{P}}\left[t^{2/a-1}\int_0^t e^{-(2\sigma/a)(B_t-B_s)}ds.\right] \\ &= t^{2/a-1}\int_0^t \mathbb{E}^{\mathbb{P}}\left[e^{-(2\sigma/a)(B_t-B_s)}\right]ds,\end{aligned}$$

which is locally integrable. In the case of $2/a < 1$, a map $x \mapsto x^{2/a}$ is concave and hence by Jensen's inequality($x \mapsto -x^{2/a}$ is convex) and Fubini's theorem,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^t e^{-\sigma(B_t-B_s)}ds\right)^{2/a}\right] &\leq \left(\mathbb{E}^{\mathbb{P}}\left[\int_0^t e^{-\sigma(B_t-B_s)}ds\right]\right)^{2/a} \\ &= \left(\int_0^t \mathbb{E}^{\mathbb{P}}\left[e^{-\sigma(B_t-B_s)}\right]ds\right)^{2/a},\end{aligned}$$

Thus, the proof ends.

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국문초록

본 논문은 Qin과 Linetsky의 확률할인인자의 장기적 분해에 대한 논의를 살펴보고 몇 가지 구체적인 예시에 대한 장기적 분해의 명확한 형태를 제시한다. 본 논문의 주요 목적은 장기적 분해가 적용될 것으로 보이는 비교적 단순한 모형들을 분석하는 것이므로, Qin과 Linetsky의 반 마팅게일 설정을 다소 간단한 1차원의 시간적으로 균등한 마르코프 설정으로 제한한다. 아직 완전히 검증되지 않은 작업 중인 몇 가지 모델들도 예시로 포함한다. Hansen과 Scheinkman이 고안한 마팅게일 추출법을 장기적 분해를 찾는 주요 도구로 소개하고 활용한다.

주요어휘: 마팅게일 추출, 장기적 분해, 확률할인인자, 양의 고유함수, 재귀 확률과정
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